

Local and Nonlocal Properties of Werner States

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We consider a special kind of mixed states – a *Werner derivative*, which is the state transformed by nonlocal unitary – local or nonlocal – operations from a Werner state. We show the followings. (i) The amount of entanglement of Werner derivatives cannot exceed that of the original Werner state. (ii) Although it is generally possible to increase the entanglement of a single copy of a Werner derivative by LQCC, the maximal possible entanglement cannot exceed the entanglement of the original Werner state. The extractable entanglement of Werner derivatives is limited by the entanglement of the original Werner state.

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Quantum entanglement plays an essential role in various types of quantum information processing, including quantum teleportation [1], superdense coding [2], quantum cryptographic key distribution [3], and quantum computation [4]. Since the best performance of such tasks requires maximally entangled states (Bell singlet states), one of the most important entanglement manipulations is the entanglement purification or distillation [5–8], namely, the process extracting maximally entangled states from input states. Most of protocols for entanglement purification (distillation) proposed so far [7,8] utilize the collective operations on many copies of a given state ρ . Strictly speaking, these protocols rely only on the properties of $\rho^{\otimes N}$ with large N and have no direct relevance to the intrinsic properties of the individual state ρ . In fact, it has been shown that there exist no purification protocols utilizing local quantum operations and classical communications (LQCC) producing a pure singlet from a *single* copy of a given mixed state of two qubits [9]. If we are not available to many copies of a given mixed state but a single one, the only task we can do by LQCC is to enhance the amount of entanglement to some extent. However, there exist entangled mixed state, for which even such a restricted task is also not successful [9–11]. Therefore, it is of fundamental importance to clarify the limit of entanglement manipulations of a single copy of a given mixed state for deeper understanding of the nature of mixed state entanglement.

We consider in this paper a special kind of mixed states – a *Werner derivative*, which refers to the state transformed by unitary – local or nonlocal – operations from a Werner state [12]. We show the followings. (i) The amount of entanglement of Werner derivatives cannot exceed that of the original Werner state. (ii) Although it is generally possible to increase the entanglement of a single copy of a Werner derivative by LQCC, the maximal possible entanglement cannot exceed the entanglement of the original Werner density matrix. The extractable entanglement of Werner derivatives is limited by the entanglement of the original Werner state. Here, the extractable entanglement of a given state ρ is referred to as the maximal possible entanglement obtained by LQCC applied to a single copy of ρ (*single-state LQCC*) [11]. The first point (i) is the direct consequence of the results presented in our recent work [13], that is, a Werner state belongs to a set of *maximally entangled mixed states*, in which the amount of entanglement cannot be increased by applying any unitary operations. The second point (ii) is our main result.

The degree of entanglement of mixed states of two qubits is customarily measured by the entanglement of formation (EOF) [8]. The EOF for a two-party pure state is defined as the von Neumann entropy of the reduced density matrix associated with one of the parties. The EOF of a bipartite mixed state is defined as $E_F(\rho) = \min \sum_i p_i E_F(|\psi_i\rangle\langle\psi_i|)$, where the minimum is taken over all possible decomposition of ρ into pure states, $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. In 2×2 systems the closed form for EOF is known [14];

$$E_F(\rho) = H\left(\frac{1 + \sqrt{1 - C^2}}{2}\right), \quad (1)$$

with $H(x) = -x \log_2 x - (1 - x) \log_2 (1 - x)$. The nonnegative real number $C = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$ is called a concurrence, where λ_i are the square roots of eigenvalues of positive matrix $\rho\tilde{\rho}$ in descending order. The spin-flipped density matrix $\tilde{\rho}$ is defined as $\tilde{\rho} = \sigma_2 \otimes \sigma_2 \rho^* \sigma_2 \otimes \sigma_2$, where asterisk denotes complex conjugation in the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ and σ_i , $i = 1, 2, 3$, are usual Pauli matrices. Since E_F is a monotonic function of C and C ranges from zero to one, the concurrence C is also a measure of entanglement.

Before verifying our main result, we firstly show that a Werner state belongs to a family of maximally entangled mixed states. Although the argument based on the convexity of concurrence is presented in Ref. [13], we follows here the direct calculations for later convenience. A Werner state in 2×2 systems takes the following form,

$$\rho_W = \frac{1-F}{3}\mathbf{I}_4 + \frac{4F-1}{3}|\Psi^-\rangle\langle\Psi^-|, \quad (2)$$

where \mathbf{I}_n denotes the $n \times n$ identity matrix and $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ the singlet state. The Werner state ρ_W is characterized by a single real parameter F called fidelity. This quantity measures the overlap of the Werner state with a Bell state. The concurrence of ρ_W is simply given by $C(\rho_W) = \max\{0, 2F-1\}$; for $F \leq 1/2$ the Werner state is unentangled, while for $1/2 < F \leq 1$ it is entangled. We assume $1/2 < F \leq 1$ so that $C(\rho_W) = 2F-1$ in the following.

The nonlocal unitary transformation, $U \in U(4)$, brings ρ_W to a new density matrix of the form,

$$\rho = \frac{1-F}{3}\mathbf{I}_4 + \frac{4F-1}{3}|\psi\rangle\langle\psi|, \quad (3)$$

where $|\psi\rangle = U|\Psi^-\rangle$. Because U preserves the rank of states, $|\psi\rangle$ is still a pure (rank of one) state vector but is generally less entangled and it can be written in a Schmidt decomposed form, $|\psi\rangle = \sqrt{a}|00\rangle + \sqrt{1-a}|11\rangle$ with $1/2 \leq a \leq 1$. The nonlocal unitary transformation U is thus parametrized by a single real number a . The Peres-Horodecki criterion (the partial transposition test) [15,16] tells us that the Werner derivative described by Eq. (3) is entangled if and only if

$$\frac{1}{2} \leq a < \frac{1}{2} \left(1 + \frac{\sqrt{3(4F^2-1)}}{4F-1} \right). \quad (4)$$

The range of parameter a is assumed to be limited by above inequalities so that ρ is always entangled. The square roots of eigenvalues of $\rho\tilde{\rho}$ are calculated as

$$\lambda_1 = \frac{(4F-1)G_+}{3}, \quad (5)$$

$$\lambda_2 = \frac{(4F-1)G_-}{3}, \quad (6)$$

and

$$\lambda_3 = \lambda_4 = \frac{1-F}{3}, \quad (7)$$

which are sorted in decreasing order. In Eqs. (5) and (6),

$$G_{\pm} = \left[2a(1-a) + G \pm 2\sqrt{a(1-a)(a(1-a) + G)} \right]^{\frac{1}{2}}, \quad (8)$$

with $G = 3F(1-F)/(4F-1)^2$. The concurrence of ρ , $C(\rho) = \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4$, is given by

$$C(\rho) = \frac{4F-1}{3}(G_+ - G_-) - \frac{2}{3}(1-F). \quad (9)$$

The problem is to find the maximal value of $C(\rho)$. We have that

$$\frac{d}{da}C(\rho) = \frac{1}{6} \frac{(4F-1)(1-2a)}{\sqrt{a(1-a)(a(1-a) + G)}} (G_+ + G_-), \quad (10)$$

which is clearly nonpositive for $a \geq 1/2$. It follows that the maximal $C(\rho)$ is achieved only for $a = 1/2$. The maximal value of the concurrence is calculated as $2F-1$. Therefore, the EOF of Werner derivatives $E_F(\rho)$ cannot exceed the EOF of the original Werner state $E_F(\rho_W)$; a Werner state is indeed a member of a set of maximally entangled mixed states.

Now let us turn to the proof of our main result. The Werner derivative ρ given by Eq. (3) can be also written as

$$\rho = \frac{1}{4}\mathbf{I}_4 + \frac{4F-1}{12} \left[(2a-1)(\mathbf{I}_2 \otimes \sigma_3 + \sigma_3 \otimes \mathbf{I}_2) + 2\sqrt{a(1-a)}(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2) + \sigma_3 \otimes \sigma_3 \right]. \quad (11)$$

Since the coefficient vectors of $\mathbf{I}_2 \otimes \boldsymbol{\sigma}$ or $\boldsymbol{\sigma} \otimes \mathbf{I}_2$ are nonzero [$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$], it is possible to increase the EOF of ρ by a single-state LQCC [11]. As shown below, however, the maximum EOF thus obtained is still less than or equal to the EOF of the original Werner state. According to Theorem 3 in Ref. [11], there exist a single-state LQCC mapping ρ to a Bell diagonal state ρ' with maximal possible EOF of the form,

$$\rho' = \frac{1}{4} \left(\mathbf{I}_4 + \sum_{i=1}^3 r_i \sigma_i \otimes \sigma_i \right), \quad (12)$$

with $r_1 \leq r_2 \leq r_3 \leq 0$. The square roots of eigenvalues of $\rho' \tilde{\rho}'$ in descending order are $\lambda'_1 = (1 - r_1 - r_2 - r_3)/4$, $\lambda'_2 = (1 - r_1 + r_2 + r_3)/4$, $\lambda'_3 = (1 + r_1 - r_2 + r_3)/4$, and $\lambda'_4 = (1 + r_1 + r_2 - r_3)/4$. Since the ratio λ'_i/λ'_j are invariant under LQCC, $\lambda'_i/\lambda'_4 = \lambda_i/\lambda_4$ ($i = 1, 2, 3$), where λ_i are given by Eqs. (5), (6), and (7). Therefore, the concurrence of ρ' , $C(\rho') = \lambda'_1 - \lambda'_2 - \lambda'_3 - \lambda'_4$, can be expressed in terms of λ_i as follows,

$$C(\rho') = \frac{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4}{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4}. \quad (13)$$

Inserting the explicit forms of λ_i into this equation, we obtain

$$C(\rho') - C(\rho_W) = 2 \frac{(1-F)G_+ - FG_- - 2(1-F)/(4F-1)}{G_+ + G_- + 2(1-F)/(4F-1)}. \quad (14)$$

The denominator of the right hand side of this equation is strictly positive so that it suffices to verify the numerator is less than or equal to zero in order to show $C(\rho') \leq C(\rho_W)$. We have that

$$\frac{d}{da} [(1-F)G_+ - FG_-] = \frac{1}{2} \frac{1-2a}{\sqrt{a(1-a)(a(1-a)+G)}} [(1-F)G_+ + FG_-], \quad (15)$$

which is clearly nonpositive for $a \geq 1/2$. It follows that maximal value of $[(1-F)G_+ - FG_-]$ is achieved for $a = 1/2$ and it turns out to be $2(1-F)/(4F-1)$. Therefore, numerator in the right hand side of Eq. (15) is strictly less than or equal to zero. Hence $C(\rho') \leq C(\rho_W)$ so that $E_F(\rho') \leq E_F(\rho_W)$. It should be noted that unitary transformation with $a = 1/2$ is just a local unitary transformation; $|0\rangle_A \rightarrow |0\rangle_A$, $|1\rangle_A \rightarrow |1\rangle_A$, $|0\rangle_B \rightarrow -|1\rangle_B$, and $|1\rangle_B \rightarrow |0\rangle_B$ such that $|\Psi^-\rangle_{AB} = (|01\rangle_{AB} - |10\rangle_{AB})/\sqrt{2} \rightarrow (|00\rangle_{AB} + |11\rangle_{AB})/\sqrt{2}$. The state ρ is, therefore, equivalent to ρ_W up to local unitary transformations and the present result is reduced to that of Ref. [10]. It implies the following. If we bring a Werner state ρ_W to one of Werner derivatives ρ by *essentially nonlocal* unitary transformations, the extractable entanglement of ρ is strictly below the EOF of the original Werner state. Our main result can be also stated in other words that the EOF of a Werner state cannot be increased by a single-state LQCC followed by nonlocal unitary transformations. This property is unique to Werner states, as shown below. If another state ρ which does not belong to a family of Werner states has the property stated above, it must be one of the maximally entangled mixed states; otherwise $E_F(\rho)$ could be increased by nonlocal unitary transformation. The maximally entangled mixed states take the following form [13],

$$\rho = p_1 |\Psi^-\rangle \langle \Psi^-| + p_2 |00\rangle \langle 00| + p_3 |\Psi^+\rangle \langle \Psi^+| + p_4 |11\rangle \langle 11|, \quad (16)$$

where $|\Psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$ and p_i are eigenvalues of ρ in decreasing order ($p_1 \geq p_2 \geq p_3 \geq p_4 \geq 0$). The state ρ can be also written as

$$\rho = \frac{1}{4} [\mathbf{I}_4 + (p_2 - p_4)(\mathbf{I}_2 \otimes \sigma_3 + \sigma_3 \otimes \mathbf{I}_2) - (p_1 - p_3)(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2) - (p_1 - p_2 + p_3 - p_4)\sigma_3 \otimes \sigma_3]. \quad (17)$$

If $p_2 \neq p_4$, the coefficient vectors of $\mathbf{I}_2 \otimes \boldsymbol{\sigma}$ or $\boldsymbol{\sigma} \otimes \mathbf{I}_2$ are nonzero and the EOF of ρ can be increased further by a single-state LQCC, which contradicts the assumed property of ρ . Therefore, the equality $p_2 = p_4$ must hold, which implies $p_2 = p_3 = p_4 = (1 - p_1)/3$. It follows that the state ρ takes the form,

$$\rho = \frac{1-p_1}{3} \mathbf{I}_4 + \frac{4p_1-1}{3} |\Psi^-\rangle \langle \Psi^-|. \quad (18)$$

Hence, ρ must be a Werner state.

Finally, we mention that Eq. (13) gives the general expression for the extractable entanglement of a given entangled state ρ of two qubits. It has the form of the concurrence of ρ , $C(\rho) = \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4$, modified by an enhancement

factor $(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)^{-1}$. For Bell diagonal states, including Werner states, $\rho = \tilde{\rho}$ so that the square roots of eigenvalues of $\rho\tilde{\rho}$ are same as the eigenvalues of ρ . Therefore, $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$ and the enhancement factor is one. It follows directly that we cannot extract higher EOF from a Bell diagonal state by a single-state LQCC as argued in Ref. [11]. For pure states of rank one, $C(\rho') = 1$, which indicates that it is always possible to extract a Bell singlet state as expected.

In summary, combined the present results with previously obtained ones [10,13], the following peculiar property of a Werner state of two qubits has been revealed; its EOF cannot be increased (i) by LQCC, (ii) by nonlocal unitary transformations, and (iii) by LQCC followed by nonlocal unitary transformations. We hope that our results presented in this paper would lead to a proper classification of entangled mixed states.

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